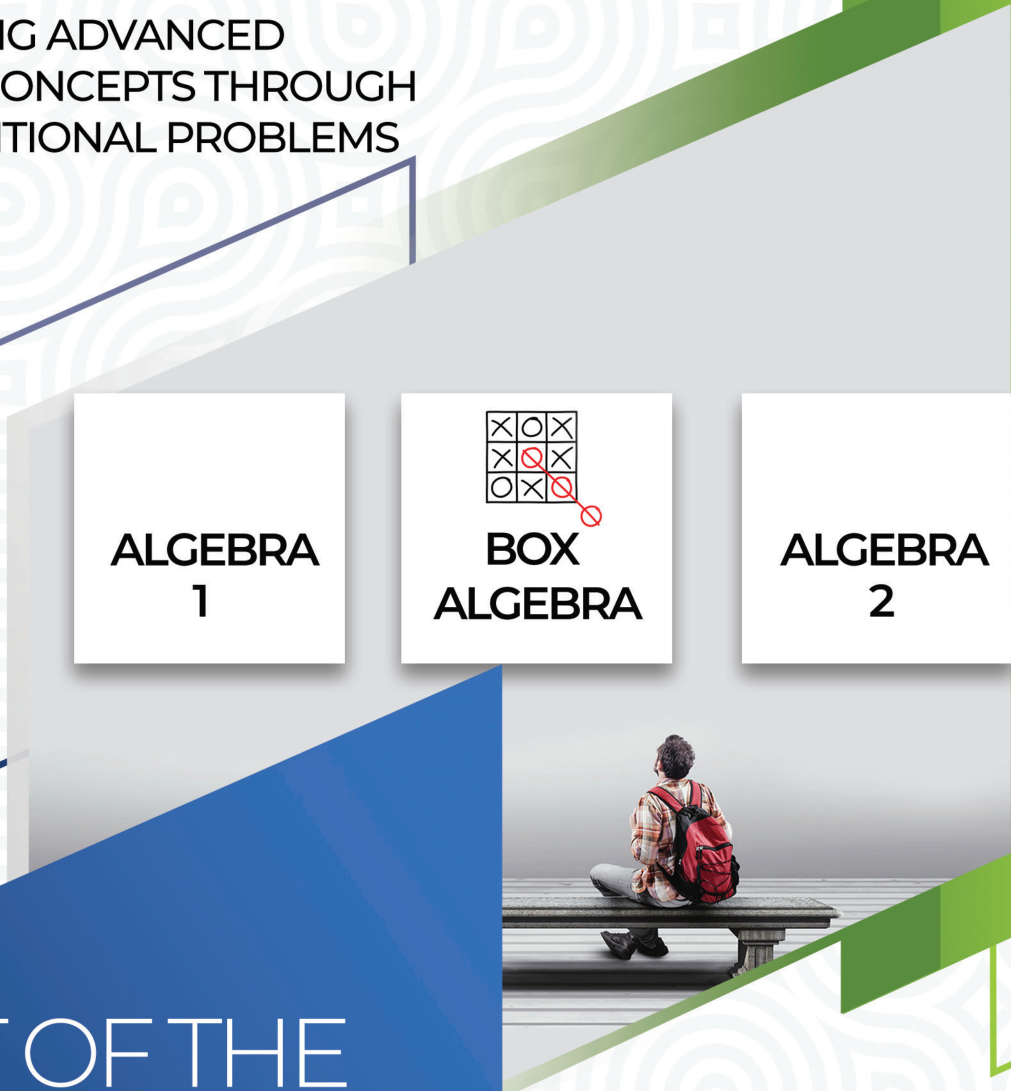
ART OF ALGEBRA

DEVELOPING ADVANCED ALGEBRA CONCEPTS THROUGH NON-TRADITIONAL PROBLEMS



OUT OF THE BOX ALGEBRA

- ADVANCED ALGERBA COURSES
- MATHEMATICS CONTESTS
- AMC 10/12
- MATHCON
- MATH LEAGUES

MATH TOPIA PRESS

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MATHEMATICS EDUCATION

PREFACE

What is BOX Algebra and what makes this book special and unique?

This book grew from working with algebra honor course students, math contest participants, and problem-solving communities. It covers all algebra 1 and most algebra 2 and pre-calculus topics and presents some non-traditional school algebra concepts. Through short explanations and carefully crafted problems, it shows how all algebraic concepts are well-connected. Working through cognitively challenging problems and their well-presented solutions, students will expand their understanding and develop advanced mathematical concepts. Students will feel challenged but eventually feel connected and enjoy the book's problems.

Possible target audiences of this book

- Algebra 1&2, honor algebra, and pre-calculus course takers.
- Participants in mathematics contests such as AMC 8, AMC 10, AMC 12, MATHCOUNTS, MathCON, and Math Leagues.
- Math Circle students and organizers.
- Pre-service and in-service mathematics teachers.
- Parents of mathematically promising students.
- General math enthusiasts.

How to use this book

Students should engage each chapter's concepts by trying themselves first, then move to examples as they work through them. Solutions are presented very clearly; however, algebra learners are supposed to explore and discover themselves when they first engage with the problems. Examples and set problems have multiple entry points and solutions that every algebra learner can begin with. There is no time pressure; students should enjoy their deep algebraic thinking over speed. Notetaking is an essential skill for algebra learners. We encourage students to take their notes and become good mathematical note-takers.

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CHAPTER 1

NUMBERS

Order of Operations

Consider the expression $5+3\times4$. Should you add 5 and 3 first, then multiply by 4? If so you get 32. Or, should you multiply 3 and 4 first, then add 5? If so you get 17. Let's take a look at the following three situations:

1. I have 3 packs of 10 pencils plus 7 extra.
   So I do: $3\times10+7 = 37$.
2. It's like 4 packs of 10 pencils with three missing.
   So I do: $4\times10-3 = 37$.

Or,

3. I have 7 pencils plus 4 packs of 10 pencils,
   so I do: $7+4\times10 = 47$.

In the final example, it does not make sense to add 7 and 4 first.

Remark 1.1

First of all, parentheses must be performed (but some exceptions) and exponents are next. Division or multiplication have the same priority, and addition or subtraction have the same priority. Again, we need to use the following conventions on order of operations.

- All powers (exponents) are considered first
- Multiplication and division are considered from left to right.
- Addition and subtraction are considered from left to right.

Problem 1.1

Evaluate the expression
$5-3\times4^3\div(7-1)$

Solution 1.1

$$\begin{aligned}5-3\times4^3\div(7-1) &= 5-3\times4^3\div6\\ &= 5-192\div6\\ &= 5-32\\ &= -27\end{aligned}$$

Problem 1.2

Simplify $2+2\times2-2\div2+2$

Solution 1.1

$$\begin{aligned}2+2\times2-2\div2+2 &= 2+4-1+2\\ &= 6-1+2\\ &= 5+2\\ &= 7\end{aligned}$$

Problem 1.3

Simplify $(3^3-3)^2\div4\times6\div1$.

Solution 1.3

$$\begin{aligned}(27-3)^2\div4\times6\div1 &= 576\div4\times6\div1\\ &= 144\times6\div1\\ &= 864\end{aligned}$$

Problem 1.4

Use parentheses to obtain 4 from the expression

$4\times4-4\div4+4$

Solution 1.4

$4\times(4-4)\div4+4 = 4$

Problem 1.5

To make the statement $(10 \ ? \ 5)+4-(10-9) = 5$ true, the question mark between 10 and 5 should be replaced by

A) $\times$ B) $-$ C) $+$ D) $\div$ E) None

Solution 1.5

$(10\div5)+4-(10-9) = 2+4-1 = 5.$

The answer is $\div$.

Remark 1.2

Holistic thinking for numbers, structures and operations. We need to look for operations holistically before deciding on whether to use one of the order of operations mnemonic (PEMDAS or BOMDAS). For example, $24\times12-23\times12$, we see that students just memorize the PEMDAS mnemonic they learned could start with multiplication and then subtraction which takes additional step. Instead, knowing appropriate number properties such as the associative property of multiplication or, the distributive property of multiplication over addition or subtraction would definitely be more efficient for comprehensive understanding of numbers and operations.

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Example 1.1

Simplify $555(143-88)-45\times6-549(143-88)$.

Notice that this expression can be though of as $(143-88)(555-549)-45\times6 = 6(143-88)-45\times6 = 6(143-88-45) = 6\times10 = 60$.

Problem 1.6

Simplify the following:

a) $4\times128\times\left(\frac{1}{4}\right)$

b) $81\times5^2-71\times5^2$

c) $25\times(13\times4)$

Solution 1.6

a) Using associative and commutative properties of multiplication,

$$\begin{aligned}4\times\left(128\times\frac{1}{4}\right) &= 4\times\left(\frac{1}{4}\times128\right)\\ &= \left(4\times\frac{1}{4}\right)\times128\\ &=1\times128\\ &=128\end{aligned}$$

b) Using the distributive property of multiplication over subtraction

$$\begin{aligned}81\times5^2-71\times5^2 &= (81-71)\times5^2\\ &= 10\times25\\ &= 250\end{aligned}$$

c) Using associative and commutative properties of multiplication

$$\begin{aligned}25\times(13\times4) &= 25\times(4\times13)\\ &= (25\times4)\times13\\ &= 100\times13\\ &= 1300\end{aligned}$$

Definition 1.1 — Natural Numbers.

$\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ The natural numbers are also called the counting numbers.

Remark 1.6

Multiplication rules of same or dif-ferent signs.

a) (+)(+) = (+) and (−)(−) = (+).

b) (+)(−) = (−) and (−)(+) = (−).

Gauss Sums

The sum of consecutive positive integers up to n is given by

$$1+2+3+...+n = \frac{n(n+1)}{2}.$$

This formula is also called "Gauss Sums".

Problem 1.17

Find the sum of 11+12+...+99+100.

Solution 1.17

$$\begin{aligned} 11+12+...+100 &= (1+2+...+100) \\ &\quad -(1+2+...+10) \\ &= \frac{100 \cdot 101}{2} - \frac{10 \cdot 11}{2} \\ &= 5050-55 = 4995 \end{aligned}$$

Problem 1.18

Find the sum of 2+4+6+...+50.

Solution 1.18

$$\begin{aligned} 2+4+6+...+50 &= 2 \cdot 1+2+...+2 \cdot 25 \\ &= 2(1+2+...+25) \\ &= 2 \cdot \frac{25 \cdot 26}{2} \\ &= 650 \end{aligned}$$

Remark 1.7

Remark Sum of Even/Odd Numbers.

$2+4+6+...+2n = n(n+1)$

$1+3+5+...+(2n-1) = n^2$

Number of integers between two integers

If a and b are integers with $a < b$, then there are $(b-a)-1$ numbers between numbers a and b, not including a or b and there are $(b-a)+1$ numbers including a and b.

Example 1.5

There are $(77-13)-1 = 63$ integers between 13 and 77, not including 13 and 77.

Example 1.6

Find the number of integers between 1 and 1000 (exclusive) that are divisible by 3. Those numbers are 3, 6, . . . ,333 and they have the same number of integers with 1, 2, . . . , 999. So, there are 333 numbers between 1 and 1000 are divisible by 3.

Problem 1.19

How many numbers between 18 and 792 are divisible by 6 (inclusive)?

Problem 1.20

Compute 21 + 23 + 25 + . . . + 99.

1. Simplify $80 - 64 \div 8 \times 4$.

A) –16 B) 0 C) 32 D) 48 E) 78

2. Simplify $16 - 4 \div (1 \div 4) + 1$.

A) –4 B) 0 C) 1 D) 4 E) 5

3. Compute the following and find the sum of all digit:

$7346 \times 1425 + 2654 \times 1425$.

A) 10 B) 12 C) 16 D) 18 E) 20

4. Simplify $6 + 3(8 - 3) \div 5 - 2^3$.

A) –2 B) –1 C) 0 D) 1 E) 2

5. Evan uses only the whole number 1–9, no more than one time each in order to obtain the smallest number by filling the following boxes

$$\overset{A}{\square}\overset{B}{\square} \times \overset{C}{\square}\overset{D}{\square}.$$

What's the difference between two numbers?

A) 7 B) 8 C) 9 D) 10 E) 11

6. Let X and Y be the following sum of sequence

$X = 20 + 22 + 24 + \dots + 100$

and

$Y = 22+24+ \dots +102$

What is the value of $Y - X$?

A) 82 B) 102 C) 122 D) 162 E) 202

7. What is the largest value of the expression $a \times b^c$ when a, b, and c are replaced with 2, 3, and 4 using each number once?

A) 36 B) 48 C) 64 D) 128 E) 162

8. Consider the following student work:
$84 \times 45 = 20 + 400 + 160 + 320$.
Which of the following is correct?

A) There is an error with 20
B) There is an error with 160
C) There is an error with 320
D) There is an error with 400
E) There is no error

9. Simplify $2.5 \times 1.8 + 2.5 \times 8.2$.

A) 15 B) 17.5 C) 18.5 D) 22.5 E) 25

10. How many terms are in the following sequences 13, 16, 19, ..., 79, 82, 85?

A) 20 B) 21 C) 24 D) 25 E) 31

11. What is the largest value of the expression $a + b^c$ when a, b, and c are replaced with 2, 3, and 4 using each number once?

A) 48 B) 66 C) 72 D) 83 E) 625

12. How many ordered pair of positive integers (m, n) that satisfy $n + m^2 \leq 17$?

A) 12 B) 34 C) 38 C) 42 E) 51

1. How many positive integers less than 10 can be obtained by using 1, 2, 3, 4 only once while four basic operations and parentheses are allowed?

A) 4 B) 6 C) 7 D) 8 E) 9

2. Given the equation

AB + CD = EFG (EFG is a 3-digit number,

AB and CD are 2-digit numbers)

place each of the digits 0, 1, 2, 3, 4, 5 and 6 in one of the letters.

What is EFG?

A) 102 B) 104 C) 105 D) 106 E) None

3. Suppose a and b are integers and a + b is an odd number. Which of the following is always true?

I. a – 2b is even

II. ab is even

III. 4a + b is even

A) Only I B) Only II C) Only III D) I and II E) I, II and III

4. Let A, B and C be non-negative integers such that $A + B + C = 8$. What is the maximum value of $A \times B \times C + A \times B + A \times C + B \times C$?

A) 36 B) 39 C) 42 D) 45 E) 48

5. If $a, b, c \in Z^+$, $a \cdot b = 13$ and $b \cdot c = 12$, then what is the value of $a + b + c$?

A) 23 B) 24 C) 25 D) 26 E) 27

6. If a, b are distinct digits, then what is the maximum value of $7a + 5b$?

A) 101 B) 103 C) 107 D) 110 E) 117

Problem Set 1

1. Multiplication and division operations have priority over addition and substraction. If multiplication and division operations are used in a single expression, the left-most operation (multiplication or division) is dealt with first. Therefore,

$80 - 64 \div 8 \times 4 = 80 - 8 \times 4$

$= 80 - 32 = 48$

The answer is D

2. For any a and b integers ($b \neq 0$),

$a \div (1 \div b) = a \times b$. Therefore,

$16 - 4 \div (1 \div 4) + 1 = 16 - 4 \times 4 + 1$

$= 16 - 16 + 1 = 1$

The answer is C

3. Multiplication has distribution property over addition. Hence,

$7346 \times 1425 + 2654 \times 1425$

$= (7346 + 2654) \times 1425 = 10000 \times 1425$

$= 14250000$

The sum of all digit of 14250000 is

$1 + 4 + 2 + 5 + 0 + 0 + 0 + 0 = 12$.

The answer is B

4. Parentheses have priority over other mathematical operations. Therefore,

$6 + 3(8 - 3) \div 5 - 2^3 = 6 + 3 \times 5 \div 5 - 8$

$= 6 + 15 \div 5 - 8 = 6 + 3 - 8 = 1$

The answer is D

5. To obtain the smallest number, we should use only 1, 2,3 and 4. $AB \times CD$ can be

$13 \times 24 = 312$, $14 \times 23 = 322$ or $12 \times 34 = 408$. We do not need to compute other possibilities since they are clearly not the smallest number can be obtained

(for example, $31 \times 24 > 13 \times 24$).

Hence, two numbers are 24 and 13, and

$24 - 13 = 11$.

The answer is E

6. Let's define $Z = 22 + 24 + \cdots + 100$. It can be seen that $X = 20 + Z$ and $Y = 102 + Z$. Hence,

$Y - X = (102 + Z) - (20 + Z) = 102 - 20 = 82$

The answer is A

7. Since all numbers are positive integers, we should choose b and c greater than a to obtain the largest value. So, there are two possibilities,

$2 \times 3^4 = 162$ or $2 \times 4^3 = 128$.

The largest value is 162.

The answer is E

8. We can write 84 as 80 + 4 and 45 as 40 + 5.

$84 \times 45 = (80 + 4) \times (40 + 5)$

$= 3200 + 400 + 160 + 20$

Therefore, 320 in the given equation should have been 3200.

The answer is C

9. Multiplication has distribution property over addition. Therefore,

$2.5 \times 1.8 + 2.5 \times 8.2 = 2.5 \times (1.8 + 8.2)$

$= 2.5 \times 10 = 25$

The answer is E

10. There is a formula for number of terms of an arithmetic sequence as follows

Number of terms =

$$\frac{\text{Last term Firs term}}{\text{Common difference of successive members}} + 1$$

Therefore, number of terms is

$\frac{85 - 13}{3} + 1 = 25$.

The answer is D

11. We should choose b and c greater than a to obtain the largest value because power functions are increasing faster than linear functions.

So, there are two possibilities,

$2 + 3^4 = 83$ or $2 + 4^3 = 66$.

The largest value is 83.

The answer is D

12. Since n is a positive integer, $n \geq 1$, and thus, $16 \geq m^2$. So, m can be 1, 2, 3 or 4.

If $m = 1$, $n \leq 16$, and there are 16 pairs of (m, n).

If $m = 2$, $n \leq 13$, and there are 13 pairs of (m, n).

If $m = 3$, $n \leq 8$, and there are 8 pairs of (m, n).

If $m = 4$, $n \leq 1$, and there are 1 pairs of (m, n).

In total, there are 38 pairs of positive integers (m, n).

The answer is C

Problem Set 2

1. There are two ways to approach to this question. In both cases, we can obtain all 1, 2, . . . , 9 numbers. If it is allowed not to use all 1, 2, 3, 4 numbers, $1 = 1$, $2 = 2$, $3 = 3$, $4 = 4$, $5 = 2 + 3$, $6 = 2 \times 3$, $7 = 3 + 4$, $8 = 2 \times 4$, and $9 = 2 + 3 + 4$.

If all numbers have to be used,

$1 = (4 - 3) \times (2 - 1)$, $2 = 1 + 2 + 3 - 4$,

$3 = (4 - 1) \times (3 - 2)$, $4 = 1 + 2 - 3 + 4$,

$5 = (4 + 1) \times (3 - 2)$, $6 = 1 - 2 + 3 + 4$,

$7 = (4 + 3) \times (2 - 1)$, $8 = -1 + 2 + 3 + 4$, and

$9 = 4 + 3 + (2 \div 1)$.

Therefore, there are 9 positive integers can be obtained.

The answer is E

CHAPTER 2

RATIONAL NUMBERS

Definition 2.1 — Rational Numbers.

A rational number is any number that can be expressed as the quotient or fraction p/q of two integers, a numerator p and a non-zero denominator q. Since q may be equal to 1, every integer is a rational number. The rational numbers are designated by $\mathbb{Q}$.

Remark 2.1

For integers a and b,

- $\frac{0}{b} = 0$ when $b \neq 0$, and $\frac{b}{0}$ is undefined otherwise.
- If the numerator of a fraction is smaller than its denominator, it is called a proper (or simple) fraction. If $\frac{a}{b}$ is simple fraction, then $-1 < \frac{a}{b} < 1$
- If the numerator of a fraction is greater than or equal to its denominator, it is called an improper fraction. If $\frac{a}{b}$ is improper fraction, then $\frac{a}{b} \leq -1$ or $\frac{a}{b} \geq 1$
- A number which consists of a whole number and a proper fraction is called a mixed number.
  $a\frac{b}{c} = a + \frac{b}{c} = \frac{a \cdot c + b}{c}$
- Let a, b, k be natural numbers with $b \neq 0$ and $k \neq 0$. Then $\frac{ak}{bk} = \frac{a}{b}$

Problem 2.1

Find a fraction with denominator 120 equivalent to $\frac{13}{24}$.

Solution

Observe that $120 \div 24 = 5$. Thus

$\frac{13}{24} = \frac{13 \cdot 5}{24 \cdot 5} = \frac{65}{120}$.

Problem 2.2

Order the rational numbers, $\frac{6}{7}, \frac{1}{7}, \frac{5}{7}, \frac{9}{7}$ from largest to smallest.

Solution

If all the rational numbers have a common denominator, then the rational number with the biggest numerator is the biggest number,

$\frac{9}{7} > \frac{6}{7} > \frac{5}{7} > \frac{1}{7}$

Problem 2.3

Compare and order the following rational numbers.

$\frac{4}{7}, \frac{5}{9}, \frac{2}{3}, \frac{10}{17}$ from

Solution

Let us equalize the numerators.

- Find the least common multiple of the numerators: LCM(4, 5, 2, 10) = 20
- Equalize the numerators:
$$\frac{4\cdot5}{7\cdot5}=\frac{20}{35},\ \frac{5\cdot4}{9\cdot4}=\frac{20}{36},\ \frac{2\cdot10}{3\cdot10}=\frac{20}{30},\ \frac{10\cdot2}{17\cdot2}=\frac{20}{34}$$
- Compare the numbers:
$$\frac{20}{30}>\frac{20}{34}>\frac{20}{35}>\frac{20}{36}\ \Rightarrow\ \frac{2}{3},\ \frac{10}{17},\ \frac{4}{7},\ \frac{5}{9}$$

Problem 2.4

Order the rational numbers:
$$\frac{11}{5},\ \frac{11}{13},\ \frac{11}{3},\ \frac{11}{12}$$

Solution

If all the rational numbers have a common numerator, then the rational number with the smallest denominator is the biggest number:
$$\frac{11}{3}>\frac{11}{5}>\frac{11}{12}>\frac{11}{13}$$

Remark 2.2

Adding or Subtracting rational numbers with different denominators, first we equalize the denominators by enlarging each rational number by the lowest common denominator (LCD). Then we add or subtract the numerators.

Problem 2.5

Add: $\frac{3}{5}+\frac{4}{7}$.

Solution

A common denominator is $5\cdot7=35$.
We thus find
$$\frac{3}{5}+\frac{4}{7}+\frac{3\cdot7}{5\cdot7}+\frac{4\cdot5}{7\cdot5}=\frac{21}{35}+\frac{20}{35}=\frac{41}{35}$$

Example 2.1

To perform the addition $\frac{2}{7}+\frac{1}{5}+\frac{3}{2}$, observe that $7\cdot5\cdot2=70$ is a common denominator. Thus
$$\begin{aligned}\frac{2}{7}+\frac{1}{5}+\frac{3}{2}&=\frac{2\cdot10}{7\cdot10}+\frac{1\cdot14}{5\cdot14}+\frac{3\cdot35}{2\cdot35}\\&=\frac{20}{70}+\frac{14}{70}+\frac{105}{70}\\&=\frac{20+14+105}{70}\\&=\frac{139}{70}.\end{aligned}$$

Remark 2.3

Multiplication of Fractions

Let a, b, c, d be natural numbers with $b\neq0$ and $d\neq0$.

Then $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

Example 2.2

We have
$$\frac{2}{3}\cdot\frac{3}{7}=\frac{6}{21}=\frac{2}{7},$$
Alternatively, we could have cancelled the common factors, as follows,
$$\frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{7}=\frac{2}{7},$$

Problem 2.6

Find the exact value of the product
$$\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{99}\right)\left(1-\frac{2}{101}\right)$$

Solution

We have,
$$\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{99}\right)\left(1-\frac{2}{101}\right)$$
$$\frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdot\frac{9}{11}\cdots\frac{97}{99}\cdot\frac{99}{101}=\frac{3}{101}$$