100% HIGH SCHOOL MATHEMATICS CHALLENGE

10 PRACTICE TESTS
WITH FULL DETAILED SOLUTIONS

Developing Math Concepts
Through Math Contest Preparation

American Mathematics Contest

AMC 10/12 - MathCON

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Problem 3. A restaurant offers four appetizers, five entrées, and three desserts. Leah will order one entrée, at most one appetizer, and at most one dessert. How many different meal combinations can Leah order?

A) 12 B) 60 C) 75 D) 90 E) 100

Problem 4. How many of the following five shapes could be the shape of the region where two triangles overlap?

I. *equilateral triangle* II. *regular pentagon* III. *regular hexagon*

IV. *square* V. *kite*

A) 1 B) 2 C) 3 D) 4 E) 5

Problem 5. Five students stand in line from shortest to tallest. The average height of the three shortest students is 58 inches, and the average height of the three tallest students is 70 inches. The average height of all five students is 63 inches. What is the median height of the five students, in inches?

A) 59 B) 64 C) 67 D) 68 E) 69

Problem 15. Five soccer teams participate in a tournament. Each team plays all the other teams exactly once. The winning team of each game is awarded 3 points and the losing team is awarded no points. If a game ends in a tie, both teams get 1 point. At the end of the tournament, four of the five teams were awarded 1, 2, 5, and 8 points in total. How many points were awarded to the fifth team?

A) 10 B) 12 C) 14 D) 15 E) None of the preceding

Problem 16. How many non-congruent quadrilaterals $ABCD$ can be constructed such that $\angle DAB = 30^\circ$, $\angle BCD = 90^\circ$, $AB = 17$, $BC = 13$ and $CD = 11$?

A) 0 B) 1 C) 2 D) 3 E) Infinitely many

Problem 17. A parabola with equation $y = ax^2 + bx + c$ has vertex (h, k). How many of the six quantities a, b, c, h, k and $\Delta = b^2 - 4ac$ can be negative at the same time?

A) At most 2 B) At most 3 C) At most 4 D) At most 5 E) All six

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Problem 21. Suppose a and b are real numbers such that $ab^2 = 1$ and $a^3 + 3b^3 = 4$. What is the product of all possible values of $a^3 + b^3$?

A) 12 B) 18 C) 24 D) 36 E) 72

Problem 22. The sum of two positive integers a and b is 2020. The number a has exactly nine positive divisors and the number b has exactly 21 positive divisors. What is $a - b$?

A) -1508 B) -868 C) 868 D) 1508 E) None of the preceding

Problem 23. For all integers $i \geq 0$, let P_i denote the point $\left(2^i, \frac{2020}{2^i}\right)$ in the xy-plane. What is the smallest integer n such that the area of convex polygon $P_0P_1P_2 \ldots P_n$ is greater than 2020^2?

A) 10 B) 11 C) 12 D) 13 E) 16

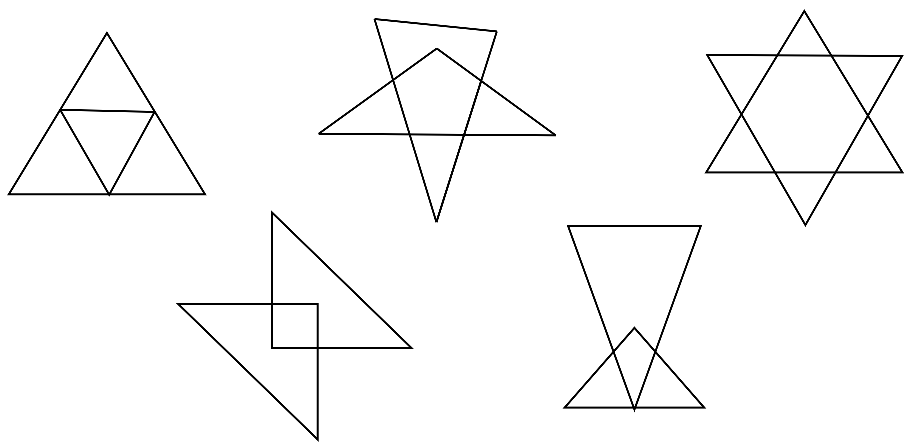
1. A
2. D
3. E
4. E
5. E
6. D
7. D
8. D
9. C
10. C
11. D
12. A
13. D
14. B
15. A
16. B
17. E
18. C
19. A
20. B
21. C
22. C
23. C
24. E
25. D

1. Factor the left-hand side as $2^{2015}(2^3 - 2^2 - 2^1 + 2^0) = 2^{2015}(3)$. Hence $a = \boxed{3}$. $\square$

2. We can write the number as $(5^3)^4 \times (2^6)^2 = 5^{12} \times 2^{12} = 10^{12}$, which has $\boxed{13}$ digits. $\square$

3. There are five choices for the entrée. There are five choices for the appetizer (either order one of four appetizers, or none at all), and four choices for the dessert (either order one of three desserts, or none at all). By the multiplication rule, the number of meal combinations is $5 \times 5 \times 4 = \boxed{100}$. $\square$

4. All $\boxed{5}$ shapes are possible:



$\square$

5. Let $a \le b \le c \le d \le e$ be the heights of the five students. We are given $a+b+c = 3\times 58 = 174$, $c+d+e = 3\times 70 = 210$, and $a+b+c+d+e = 5\times 63 = 315$. Adding the first two equations and subtracting the third, we obtain $c = 174 + 210 - 315 = 69$, so the median height is $\boxed{69}$ inches. $\square$

6. We have $|n^2 - 6n + 5| = |(n-5)(n-1)|$, so in order for this expression to be prime, either $n - 5$ or $n - 1$ should be ± 1, and the other should be $\pm p$ for some prime p. Testing, we see that $n = 0$, $n = 2$, $n = 4$, and $n = 6$ all work, so the answer is $\boxed{4}$. $\square$

7. We want the probability of obtaining 3 or more heads. The probability is

$$\frac{\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}}{2^6} = \frac{20 + 15 + 6 + 1}{64} = \boxed{\frac{21}{32}}$$

. $\square$

8. Suppose the edge lengths are a, $\frac{3}{2}a$, and $2a$, where a is an integer. Since $\frac{3}{2}a$ is an integer, a must be even, so let $a = 2k$ for some positive integer k. Then the dimensions are $2k$, $3k$, and $4k$, and the volume is $24k^3$. The only answer choice of the form $24k^3$ for some integer k is $\boxed{192}$. $\square$

9. Let the terms of the first sequence be 1, $1+x$, $1+2x$, …, $1+19x$, and let the terms of the second sequence be 10, $10+y$, $10+2y$, …, $10+9y$. The sum of the terms of these two sequences are equal, so we have

$$\begin{aligned} 1+(1+x)+(1+2x)+\ldots+(1+19x) &= 10+(10+y)+(10+2y)+\ldots+(10+9y) \\ 20+\frac{19\cdot 20}{2}x &= 100+\frac{9\cdot 10}{2}y \\ 38x-9y &= 16 \end{aligned}$$

Since x and y are positive integers, we want to solve the Diophantine equation $38x-9y=16$ while minimizing $x+y$. Taking this equation modulo 9, we obtain $2x \equiv -2 \pmod 9$, so $x \equiv 8 \pmod 9$. We see that $(x, y) = (8, 32)$ is the smallest solution in positive integers, so $\min(x+y) = 8+32 = \boxed{40}$. □

10. Since $105 \times N = 100 \times N + 5 \times N$, we have

$$\underbrace{111\ldots11}_{\text{105 digits}}00+\underbrace{555\ldots55}_{\text{105 digits}} = 11\underbrace{666\ldots66}_{\text{103 digits}}55.$$

Thus the sum of the digits of the product $105 \times N$ is

$$1+1+103\times 6+5+5=\boxed{630}.$$

□

11. With probability $\frac{1}{2^{10}}\binom{10}{i}$, Mark will flip i heads and $10-i$ tails. In this case, the number on the board will be $\frac{2^i}{2^{10-i}} = 2^{2i-10}$. Applying the definition of expectation, where X represents the number on the blackboard after 10 seconds:

$$\begin{aligned} \mathbb{E}(X) &= \sum_{i=0}^{10} \frac{1}{2^{10}}\binom{10}{i} \times 2^{2i-10} \\ &= \frac{1}{2^{20}}\sum_{i=0}^{10}\binom{10}{i}2^{2i} \\ &= \frac{1}{2^{20}}\sum_{i=0}^{10}\binom{10}{i}4^{i} \end{aligned}$$

The sum $\sum_{i=0}^{10}\binom{10}{i}4^i$ is simply the binomial expansion of $(1+4)^{10} = 5^{10}$. Hence $\mathbb{E}(X) = \frac{5^{10}}{2^{20}} = \boxed{\frac{5^{10}}{4^{10}}}$.

Alternate solution: For $i = 1, \ldots, 10$, let X_i be a random variable which equals 2 with probability $\frac{1}{2}$, and $\frac{1}{2}$ with probability $\frac{1}{2}$. Note that $\mathbb{E}(X_i) = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{1}{2} = \frac{5}{4}$. After 10 seconds, the number on the board is $X = \prod_{i=1}^{10} X_i$. Using the fact that the expected value of the product of independent random variables equals the product of the expected values, we have

$$\mathbb{E}(X) = \mathbb{E}\left(\prod_{i=1}^{10} X_i\right) = \prod_{i=1}^{10} \mathbb{E}(X_i)$$
$$= \boxed{\left(\frac{5}{4}\right)^{10}}$$

□

12. By the triangle inequality, the sum of the lengths of any two sides must be strictly larger than the third side. We see that the only possible side lengths are $(51, 51, 98)$, $(52, 52, 96)$, $(53, 53, 94)$, $\ldots$, $(99, 99, 2)$, giving $99 - 51 + 1 = \boxed{49}$ triangles. □

13. Observe that for all integers b, $2\spadesuit b = 2b - 2(2 + b) = 2b - 4 - 2b = -4$. Then $1\spadesuit(2\spadesuit(3\spadesuit(4\spadesuit 5))) = 1\spadesuit(-4) = 1(-4) - 2(1 - 4) = \boxed{2}$. □

14. Using the formula for the number of positive divisors of an integer, we observe that if $d(n) = 4$, then $n = p^3$ or $n = pq$ for distinct primes p and q.

    If either m or $m + 1$ is the cube of a prime, we see that $(7, 8)$ or $(8, 9)$ do not give us a solution, but $(26, 27)$ does, so $m = 26$ is a candidate. Otherwise, m and $m + 1$ are both the product of two distinct primes. The smallest primes are 2, 3, 5, 7; we quickly see that $2 \times 7 = 14$ and $3 \times 5 = 15$. Hence $m = 14$, so the answer is $1 + 4 = \boxed{5}$. □

15. Let W, L and T represent a win, a loss and a tie respectively. Let A, B, C and D be the four teams who gets 1, 2, 5 and 8 points in total, respectively. Let E be the fifth team.

    Neither A nor B has a win, since both of them have fewer than 3 points, i.e. no wins. So A has $\{T, L, L, L\}$ and B has $\{T, T, L, L\}$. Moreover, the match between A and B must be a tie. Thus A lost all other matches against C, D, E.

    Since C has a win against A and 5 points in total, C has $\{W, T, T, L\}$. Then the match between B and C must be a tie, since neither B nor C has no other wins. Thus B loses its matches against D and E.

    Since D has two wins against A and B, and 8 points in total, D has $\{W, W, T, T\}$, i.e. its other matches against C or E are tie.

    Finally, E has a win against A, B and C, while it has a tie against D. So its total score is $3 \cdot 3 + 1 = \boxed{10}$. □

16. Since $\angle BCD = 90°$, $BC = 13$, and $CD = 11$, we can fix the relative locations of B, C, and D. By the Pythagorean theorem, $BD = \sqrt{13^2 + 11^2} = \sqrt{290}$, so in particular, $AB < BD$. Then A must lie on the circle of radius 17 centered at B: